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Toposym 4-B

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TYCHONOFF SPACES THAT HAVE A COMPACTIFICATION
WITH COUNTABLE REMAINDER

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In 1935, L. Zippin showed that every separable rimcompact completely metrizable space has a metrizable compactification with a countable (not necessarily infinite) remainder [Z]. A Tychonoff space X with a compactification γX such that $|\gamma X - X| \leq \omega$ is called a *Zippin space* and γX is called a *Zippin compactification*. If, in addition, $\gamma X - X$ is metrizable, X is called a *strongly Zippin space* and γX a *strongly Zippin compactification*. In this paper, an attempt is made to characterize spaces that are Zippin or strongly Zippin.

We succeed in this goal only in small part, but we do obtain a number of conditions on a space that are either necessary or sufficient for such compactifications to exist. For the most part, proofs are omitted. A more complete version of this paper will appear elsewhere.

At the Fourth Prague Topological Symposium, T. Hoshina also presented a paper on this topic. His results and mine overlap, but are not identical.

All topological spaces considered are assumed to be Tychonoff spaces. Any such space has a maximal compactification βX , called the *Stone-Cech compactification* of X that maps continuously onto any compactification γX of X with a mapping that extends the identity map [GJ, Chapter 6]. If the topology of X has a base of open sets with compact boundary, then X is called *rimcompact* (the term *semicompact* is used in [Z] and *semibicompact* is used in [M]). Every rimcompact space has a compactification ΦX maximal among the compactifications with a zero-dimensional remainder. ΦX is called the *Freudenthal compactification* of X [I, pp. 109-122] [M].

If P is a property of topological spaces, then X has P at ∞ if $\beta X - X$ has P . It is noted in [HI, Sec. 3] that if P is compactness, local compactness, σ -compactness, or the Lindelöf property, then X has P at ∞ if and only if $\gamma X - X$ has P for any compactification γX of X . A space that is σ -compact at ∞ is said to be *Cech-complete* or an *absolute G_δ* . It is well known that a metrizable space is Cech-complete if and only if it admits a complete metric [E, p. 190]. X is Lindelöf at ∞ if and only if every compact subset K_1 of X is contained in a compact set K_2 for which there is a countable family $\{U_i\}$ of open sets containing K_2 such that any open set containing K_2 contains some U_i . In particular, every metrizable space is Lindelöf at ∞ [HI, Sec. 3]. Also, if X is Lindelöf at ∞ and has a compactification with 0-dimensional

remainder, then X is rimcompact by [I, p. 114].

It follows that every Zippin space is rimcompact and Čech-complete. (See also [R1] [R2]). As is noted in [I, p. 109]:

$Cl_{\gamma X}(\gamma X - x) = (\gamma X - X) \cup R(X)$ for any compactification γX of X , where $R(X)$ is the set of points of X that fail to have a compact neighborhood.

Thus, by [CN, Sec. 6], we have:

1. *Proposition* If X is a Zippin space then

- (a) X is rimcompact.
- (b) X is Čech-complete.
- (c) $|R(X)| \leq \exp \exp \omega$.

If X is strongly Zippin, then, in addition:

- (d) $R(X)$ is a Lindelöf space.

The upper bound in (c) cannot be lowered. For if Q is the space of rational numbers, then βQ is a strongly Zippin compactification of $\beta Q - Q = R(\beta Q - Q)$, and $|\beta Q| = |\beta Q - Q| = \exp \exp \omega$ [GJ, Chap. 9].

Whether the conditions of Proposition 1 are sufficient to insure that a space X is a Zippin space remains an open question. Below, two kinds of sufficient conditions are obtained; those that make $R(X)$ a "large" part of X , and those that make it in a sense "small". I begin with the former.

A space X such that every family of pairwise disjoint of open sets is countable is said to satisfy the *countable chain condition* (CCC). A space X is called *metacompact* or *weakly paracompact* if every open cover has a point-finite open refinement. As is well known, every paracompact, and hence every metrizable space is metacompact [E, pp. 225-228].

As in [LM], a space X is called *dense separable* if every dense subspace of X is separable.

2. *Theorem.* Suppose X is a Zippin space such that $X - R(X)$ is separable. Then:

- (a) X satisfies the CCC.
- (b) If X is metacompact or strongly Zippin, then X is a Lindelöf space.
- (c) If X is strongly Zippin, then X is separable.
- (d) If $X - R(X)$ is dense separable, so is X .

3. *Corollary.* Suppose X is a metrizable space such that $(X-R(X))$ is separable. Then the following are equivalent.

- (a) X is a strongly Zippin space.
- (b) X is a Zippin space.
- (c) X is separable, rimcompact, and Čech-complete.

Next, a characterization of a special class of strongly Zippin spaces is given. It is established by decomposing the remainder of X in its Freudenthal compactification ΦX .

4. *Theorem.* If $R(X)$ is locally compact, then X is a strongly Zippin space if and only if X is rimcompact, Čech-complete, and $R(X)$ is a Lindelöf space. Indeed, such a space has a strongly Zippin compactification with remainder homeomorphic to either a countable discrete space or its one-point compactification.

I conclude with some remarks, examples, and questions.

A. By modifying [LM, Example 5.3], an example can be given of a Zippin space that is not strongly Zippin. It can be shown, however, that if $R(X)$ is Lindelöf and X is a Zippin space, then X is strongly Zippin.

B. Clearly every closed subspace of a (strongly) Zippin space is (strongly) Zippin, and every open subspace of a Zippin space is rimcompact and Čech-complete by Proposition 1. The existence of open subspaces of $\beta Q - Q$ that are not Lindelöf shows that an open subspace of a strongly Zippin space need not be strongly Zippin. I do not know, however, if an open subspace of a (strongly) Zippin space has to be a Zippin space.

C. Recall that a continuous closed surjection $f: X \rightarrow Y$ such that $f^{-1}(y)$ is compact for every $y \in Y$ is called a *perfect* map. If $Y = [0,1] - Q$, then the projection map of $Y \times [0,1]$ onto Y is perfect, Y is a strongly Zippin space, but $Y \times [0,1]$ is not rimcompact and hence is not a Zippin space (although it is the product of a compact space and a strongly Zippin space). I do not know, however, if a perfect image of a (strongly) Zippin space must be (strongly) Zippin.

D. It follows easily from [GM, Example 5.3, ff.] that no connected Zippin space has a countable partition into compact sets.

E. It is easily verified that if $R(X) = X$ is connected, then the remainder of X in any compactification is connected, whence X cannot be a Zippin space. (See [R 1, Corollary 3]). Indeed, if X is also Lindelöf at ∞ , it cannot even be rimcompact. In particular, a countably infinite product of copies of R is not rimcompact.

F. It was shown by McCartney in [Mc, 3.6] that X has a maximal Zippin compactification if and only if X has a compactification with zero-dimensional remainder

and $\beta X-X$ has only countably many components. Indeed, if this latter holds, then ΦX is the maximal Zippin compactification. For a simpler proof see [D].

I am indebted to M. Rayburn for suggesting the study of Zippin spaces, and to Tom Savage and him for valuable conversations about it. Much of this material was presented at a seminar at Wesleyan University in the summer of 1976 and valuable comments were made by W. Comfort, T. Hager, L. Robertson, and T. Retta.

References

- [CN] W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer-Verlag, New York, 1974.
- [D] R. Dyckhoff, *Perfect light maps as inverse limits*, Quat. J. Math., Oxford (2) 25 (1974), 441-449.
- [E] R. Engelking, *Outline of General Topology*, North-Holland Publishing Co., Amsterdam, 1968.
- [GJ] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand Publishing Co., New York, 1960.
- [GM] J. de Groot and R. McDowell, *Locally connected spaces and their compactifications*, Ill. J. Math., 11 (1967), 353-364.
- [HI] M. Henriksen and J. Isbell, *Some properties of compactifications*. Duke Math. J., 25 (1958), 83-105.
- [I] J. Isbell, *Uniform spaces*, Amer. Math. Soc. Survey no. 12, Providence, R. I., 1964.
- [LM] R. Levy and R. McDowell, *Dense subsets of βX* , Proc. Amer. Math. Soc., 50 (1975), 426-430.
- [Mc] J. McCartney, *Maximum zero-dimensional compactifications*, Proc. Camb. Phil. Soc., 68 (1970), 653-661.
- [M] K. Morita, *On bicompatifications of semibicompact spaces*, Sci. Rep. Tokyo Bunrika Daigaku, Sec. A4, no. 92 (1952), 200-207.
- [R 1] M. Rayburn, *On the Stoilow-Kerékjártó compactification*, J. London Math. Soc., (2) 6 (1973), 193-196.
- [R 2] M. Rayburn, *On Hausdorff compactifications*, Pacific J. Math., 44 (1973), 707-714.
- [Z] L. Zippin, *On semi-compact spaces*, Amer. J. Math., 57 (1935), 327-341.